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The Consistent Criteria of Hypotheses for Gaussian Homogeneous Fields Statistical Structures

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ABSTRACT
The present theory of consistent criteria for testing hypothesis of statistical structures of homogeneous Gaussian fields can be used, for example, in the reliability prediction of different engineering designs. In the paper there are discussed Gaussian homogeneous fields statistical structures $\{E, S_i, \mu_i, i \in I\}$ in Hilbert space of measures. We define the consistent criteria for checking hypotheses such as the probability of any kind of errors is zero for given criteria. We prove necessary and sufficient conditions for the existence of such criteria.
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Let there is given (E, S) measurable space and on this space there is given $\{\mu_i, i \in I\}$ family of probability measures depended on $i \in I$ parameter.Let bring some definition $($*see* $[1]-[9])$.

Definition 1: A statistical structure is called object $\{E, S_i, \mu_i, i \in I\}$.

Definition 2: A statistical structure $\{E, S_i, \mu_i, i \in I\}$ is called orthogonal (singular) if μ_i and μ_j are orthogonal for each $i \neq j, \ i \in I, \ j \in I$.

Definition 3: A statistical structure $\{E, S_i, \mu_i, i \in I\}$ is called separable, if there exists family S-measurable sets $\{X_i, i \in I\}$ such thatthe relations are fulfilled:

1. $\forall i \in I, \ \forall j \in I \quad \mu_i(X_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$
2. $\forall i \in I, \ \forall j \in I \quad card\ (X_i \cap X_j) < c$, Where c denotes power continuum.

Definition 4: A statistical structure $\{E, S_i, \mu_i, i \in I\}$ is called weakly separable if there exists family S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled: $\mu_i(X_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

Definition 5: A statistical structure $\{E, S_i, \mu_i, i \in I\}$ is called strongly separable is There exists disjoint family S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled $\forall i \in I \quad \mu_i(X_j) = 1$.

Remark 1: A strong separable there follows weakly separable. From weakly separable there follows orthogonal but not vice versa.

Example 1: Let $E = [0,1] \times [0,1]$ and S be Borel σ -algebra of parts of E. Take the S-measurable sets

$X_i = \{(x, y) | 0 \leq x \leq 1, \quad y = i, \ i \in (0,1]\}$.

Let li be are linear Lebesgue probability measures on $X_i, \ i \in (0,1]$. and l_0 Lebesgue plane probability measures on $E = [0,1] \times [0,1]$ then the statistical structure $\{E, S_i, l_i, i \in [0,1]\}$ is orthogonal, but not weakly separable.

Remark 2: The countable family of probability measures $\{\mu_k, k \in N\}, \ N = \{1, 2, \ldots\}$ strongly separable, weakly separable, separable and orthogonal are equivalent, (see 2,3,4).

Remark 3: On an arbitrary set E of continuum power one can define Gaussian homogeneous orthogonal statistical structure having maximal possible power equal to 2^{2^c}, Gaussian homogeneous weakly statistical structure having maximal possible power equal to 2^c, where C is continuum power (see 3).

Remark 4: A.V Ckorokhod proved (in ZFC&CH theory) that continual weakly separable statistical structure as much strongly separable Z.Zerakidze proved (in ZFC&MA theory) that continuum power Borelweakly separable statistical structure as much strongly separable. We deote be (MA) the Martins axiom.

Let $|\mathrm{H}|$ be set of Hypotheses and $\mathrm{B}(|\mathrm{H}|)$ be σ - algebra of subsets of $|\mathrm{H}|$ which contains all finite subsets of $|\mathrm{H}|$.

Definition 6: A statistical structure $\{E, S, \mu_H, H \in |\mathrm{H}|\}$ will be said to admit a consistent criteria for checking hypotheses, if there exists at least one measurable map $\delta:(E,S)\to(|\mathrm{H}|,\mathrm{B}(|\mathrm{H}|))$, such that $\mu_H(x:\delta(x)=H)=1 \quad \forall H\in|\mathrm{H}|$.

Remark 5. By Z.Zerekidze was introduced definition a consistent criterion for checking hypotheses (see 2).

Definition 7. The following probability $\alpha_H(\delta)=P_H(x:\delta(x)\neq H)$ is called the probability of error of the H-th kind for a given criterion δ .

Known the following theorem (see3)

Theorem 1. The statistical structure $\{E, S, \mu_H, H \in |\mathrm{H}|\}$ admits consistent criteria for checking hypotheses if and only if the probability of error of all kind is equal to zero for the criterion δ .

Let M^{σ} be a real linear space of all alternating finite measurable on S.

Definition 8. A linear subset $M_H \subset M^{\sigma}$ is called a Hilbert space of measurable if.

(1) One can introduce on M_H a scale product $\langle \mu,\nu\rangle$, $\mu,\nu\in M_H$ such that M_H is the Hilbert space , and for every mutually singular measurable μ and ν , $\mu,\nu\in M_H$ the scale product $\langle\mu,\nu\rangle=0$;

(2) If $\nu\in M_H$ and $|f|\le 1$, then $\nu_f(A)=\int_A f(x)\nu(dx)\in M_H$ where $f(x)$ is the S-measurable real function and $\langle\nu_f,\nu_f\rangle\le\langle\nu,\nu\rangle$.

Let $t=(t_1,t_2,\ldots,t_n)\in T$, where T be a closed bounded subset of R^n, $\xi_i(t)$, $t\in T$, $i\in I$. Gaussian real homogenous field on T with zero means $E[\xi(t)]=0$, $\forall i\in I$, and correlation function $E[\xi_i(t)\xi_i(s)]=R_i(t-S)$, $t\in T$, $s\in T$, $\forall i\in I$. $\{\mu_i, i\in I\}$ be the corresponding probability measures given on S and $f_i(\lambda)$, $\lambda\in R^n$, $\forall i\in I$ be spectral densities.

We becalled the Fourier transformation of generalized function in the sense of Schwartz as generation Fourier transformation.

Let

$$\int_{R^n}\int_{R^n}\frac{\left|\tilde{b}(\lambda,\mu)\right|^2}{f_i(\lambda)f_i(\mu)}d\lambda d\mu=+\infty,\quad \forall i\in I,$$ where $\tilde{b}(\lambda,\mu)$, $\lambda,\mu\in R^n$ the Generalization Fourier transformation of the following function $b(s,t)=R_i(s,t)-R_j(s,t)$, $t\in T$, $\forall i,j\in I$.

Then μ_i and μ_j are pairwise orthogonal (see 9,10) and $\{E, S_i, \mu_i, i\in I\}$ are the Gaussian orthogonal homogenous fields statistical structures. Next, we consider S-measurable $g_i(x)$, $i\in I$ functions such that $\sum_{i\in I}\int_E|g_i(x)|^2\mu_i(dx)<+\infty$. Let M_H the set measures defined by the formula $\nu(B)=\sum_{i\in I_1}\int_B g_i(x)\mu_i(dx)$, where $I_1\subset I$ a countable subsets in I is and $\sum_{i\in I_1}\int_E|g_i(x)|^2\mu_i(dx)<\infty$. define a scale product on M_H by formula

$$\langle\mu_1,\mu_2\rangle=\sum_{i\in I_1\cap I_2}\int_E g_i^1(x)g_i^2(x)\mu_i(dx),$$

Where $\mu_j(B)=\sum_{i\in I_j}\int_B g_i^j(x)\mu_i(dx)$, $j=1,2$, $\forall B\in S$ then M_H is a Hilbert space of measures and $M_H=\bigoplus_{i\in T}M_H(\mu_i)$, where $M_H(\mu_i)$ is the Hilbert space of elements the form $\nu(B)=\sum_{i\in I_1}\int_B g_i(x)\mu_i(dx)$, $B\in S$, $\int_E|f(x)|^2\mu_i(dx)<\infty$, with the scale product on $M_H(\mu_i)$ by formula $\langle\nu_1,\nu_2\rangle=\int_E f_1(x)f_2(x)\mu_i(dx)$, where $\nu_j(B)=\int_E f_j(x)\mu_i(dx)$, $j=1,2$. (see 7).

Let $\{H_i\}$ be a countable family of hypotheses, denote $F=F(M_B)$ the set of real functions f for which $\int_E f(x)\mu_{H_i}(dx)$ is defined for all $\mu_{H_i} \in M_H$, where $M_H = \oplus_{i \in N} M_H(\mu_i)$.

Theorem 2. Let $M_H = \oplus_{i \in N} M_H(\mu_i)$ be a Hilbert space of measures. The Gaussian homogenous fields orthogonal statistical structures $\{E, S, \mu_{H_i}, i \in N\}$ admits a consistent criteria for checking hypotheses if and only if the correspondence $f \to \mu_j$ defined by the equality $\int_E f(x)\mu_H(dx) = \langle \mu_f, \mu_H \rangle, \ \forall \mu_H \in M_H$ is one –to-one , where $f \in F(M_B)$.

Proof. Sufficiency. For $f \in F(M_B)$ we define the linear functional l_f by the equality $\int f(x)\mu_H(dx) = l_f(\mu_H) = \langle \mu_f, \mu_H \rangle$. Denote as l_f a countable subset in N, for which $\int_E f(x)\mu_{H_i}(dx) = 0$ for $i \notin I_f$.

Let us consider the functional $l_{f_{H_i}}$ on $M_H(\mu_H)$ to which it corresponds Then for $\mu_{H_1}, \mu_{H_2} \in M_H(\mu_i)$ we have $\int_E f_{H_1}(x)\mu_H(dx) = \int_E f_1(x) f_2(x)\mu_{H_i}(dx) = \int_E f_{H_1}(x) f_{H_2}(dx)$.

Therefore $f_{H_1} = f_1$ a.e. with respect to the measure μ_{H_i} Let $f_{H_i} > 0$ a.e. with respect to the measures μ_{H_i} and $\int_E f_{H_i}(x)\mu_{H_i}(dx) < \infty$, $\mu_{H_i}(c) = \int_c f_{H_i}(x)\mu_{H_i}(dx)$, then $\int_E f_{H_i}(x)\mu_{H_j}(dx) = \langle \mu_{H_i}, \mu_{H_j} \rangle = 0, \ \forall i \neq j$.

Denote $C_{H_i}\{x : f_{H_i}(x) > 0\}$, then $\int_E f_{H_i}(x)\mu_{H_j}(dx) = 0, \ i \neq j$. Hence if follows, that $\mu_{H_j}(C_{H_i}) = 0, \ i \neq j$. On the other hand $\mu_{H_j}(E - C_{H_i}) = 0$. Therefore the statistical structure $\{E, S, \mu_{H_i}, i \in N\}$ is weakly separable and from Remark 2 follows that the statistical structure $\{E, S, \mu_{H_i}, i \in N\}$ is strongly separable. It is obvious that $\{C_{H_i}, i \in N\}$ is a disjunctive family of S-measurable sets and $\mu_{H_i}(C_{H_i}) = 1, \ \forall i \in N$. Let us definea mappch $(E, S) \to (H, B(H))$ Likethet $\delta(C_{H_i}) = H_i, \ \forall i \in N$. We have $\mu_{H_j}(x : \delta(x) = H_i) = 1, \ \forall i \in N$.

Necessity. Since the statistical structure $\{E, S, \mu_{H_i}, i \in N\}$ admits a consistent criterion for checking hypotheses then the statistical structure $\{E, S, \mu_{H_i}, i \in N\}$ is strongly separable (see 2,4), so there exists S-measurable sets $\{X_i, i \in N\}$ such that $\mu_{H_i}(X_i) = 1, \ \forall i \in N$.

We put the measure μ_{H_i} unto the correspondence to a function $I_{H_i} \in F(M_B)$. Then

$$\int_E I_{H_i}(x)\mu_{H_j}(dx) = \int I_{H_i}(x) \cdot I_{H_i}(x)\mu_{H_i}(dx) = \langle \mu_{H_i}, \mu_{H_i} \rangle.$$

The function $f_{H_1}(x) = f_1(x) \cdot I_{H_i}(x) \in F(M_B)$ we put the measures $\mu_{H_1} \in M_H(\mu_{H_i})$. Then

$$\int_E f_{H_1}(x)\mu_{H_i}(dx) = \int_E f_1(x) I_{H_i}(x)\mu_{H_2}(dx) = \int_E f_1(x) f_2(x) I_{H_i}(x)$$

$\forall \mu_{H_2} \in M_H(\mu_i)$. We put the measures $\nu \in M_H$, where $\nu = \sum_{i \in I_1 \in N} \int g_i(x)\mu_{H_i}(dx)$ into the correspondence to a function

$f(x) = \sum_{i \in I_1 \subset N} g_i(x) I_{H_i}(x) \in F(M_B)$. Then

$\forall \nu_1 \in M_H, \ \nu_1 = \sum_{i \in I_2 \subset N} \int g_i^1(x)\mu_{H_i}(dx)$ we have

$$\int f(x)\nu_1(dx) = \int_E \sum_{i \in I_1 \cap I_2} g_i(x) g_i^1(x)\mu_{H_i}(dx) = \sum_{i \in I_1 \cap I_2} \int g_i(x) g_i^1(x)\mu_{H_i}(dx) = \langle \nu, \nu_1 \rangle.$$

It follows from the proven theorem that the indicated above correspondence puts some functions $f \in F(M_B)$ into the correspondence to each linear continuous functional l_f. If in $F(M_B)$ we identify function considering with respect the measures $\{E, S, \mu_{H_i}, i \in N\}$ the correspondence will be bijecfive. The Theorem 2 is proved.

Example 2. Let $E = R = (-\infty;+\infty)$, S be Borel σ -algebra of parts of R and μ_{H_m} are Lévesque measure on $[m, m+1)$ $m \in Z$. The statistical structure $\{R, S, \mu_m, m \in Z\}$ is strongly separable. Let is define δ map $(R, S) \to (H, B(H))$ like that $\delta([m, m+1)) = H_m$, $m \in Z$. We have $\mu_{H_m}(\delta(x) = H_m) = 1$, $\forall m \in Z$ e. i. $\delta(x)$ is consistent criteria for checking hypotheses such as the probability of any Kind of errors is zero for given criteria δ.

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